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AN INTEGRAL EQUATION APPROACH TO HEAT AND MASS TRANSFER PROBLEM IN AN INFINITE CYLINDER

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Abstract—In this paper the authors have developed a method of integral equations to obtain the transfer potentials for heat and mass in a body of unidimensional infinite cylinder under the influence of the most general type of boundary conditions. An approximate solution, applicable for small values of generalized time has also been worked out for the system.

NOMENCLATURE

a, coefficient of diffusivity;
A_i, B_i, dimensionless known thermophysical coefficients;
c, specific content;
$f_i(x)$, given functions;
Fo, Fourier number;
r, space variable;
R, radius of the cylinder;
t, time variable;
T, thermal potential;
U, concentration of the matter;
x, dimensionless space variable;
γ, density;
λ, coefficient of conductivity;
ε, phase criterion;
ρ, specific heat of evaporation;
δ, Soret coefficient;
θ_i, dimensionless transfer potentials.

Subscripts
m, matter;
q, heat;
x, derivative with respect to x.

Superscript
0, characteristic entity.

Suffixes
i, j, s, 1, 2;
n, 1, 2, ...

INTRODUCTION

There are a number of processes in which the diffusion takes place through the pores of solid body which may absorb and immobilize some of the diffusing matter with the evolution or absorption of heat at times accompanied by heat changes due to change of state. The heat of evolution or absorption diffuses through the medium and produces a cross effect on the absorption of diffusing matter through the body. Thus the phenomenon becomes different from the individual transfer phenomenon of heat or matter and needs a simultaneous consideration of the transfer phenomena of both.

Crank [1], Luikov and Mikhailov [2] have considered a number of phenomena of this type but their considerations were limited with the simple type of the interaction law between the surface of the solid body and the medium. However, in the context of the [2] the authors have also included a general type of the interaction law for an infinite plate. They have developed an integral equation to solve the problem by considering first an auxiliary problem in which the transfer potentials are supposed to be prescribed.

In this paper, the authors have applied a similar approach as in [2] to obtain the transfer potentials for heat and matter in an infinite cylinder. In the first step, they have constructed an auxiliary problem in which the fluxes of the transfer potentials at the surface are prescribed. An approximate solution has also been worked out for small values of generalized time.

STATEMENT OF THE PROBLEM

The transfer of heat and matter in a capillary porous body is followed by the equations

$$c_q \gamma \frac{\partial T}{\partial t} = \nabla(\lambda q \nabla T) + \epsilon \rho c_m \gamma \frac{\partial U}{\partial t} \tag{1}$$

and

$$c_m \gamma \frac{\partial U}{\partial t} = \nabla(\lambda_m(\nabla U + \delta \nabla T)), \tag{2}$$

where $T = T(r, t)$ and $U = U(r, t)$ are the transfer potentials of heat and matter.

Now we shall define some dimensionless variables:

$x = r/R,\ Fo = a_q t/R^2,\ \theta_1 = T/T^0,\ \theta_2 = U/U^0$

and the similarity criteria:

(i) the Luikov criterion of the field of bound matter in relation to the temperature field

$$Lu = a_m/a_q,$$

(ii) the Posnov criterion for bound matter

$$Pn = \delta_m T^0/U^0$$

(iii) the Kossovich criterion for bound matter

$$Ko = \rho U^0/c_q T^0.$$

The equations (1) and (2) can be written in the dimensionless form with the aid of the dimenless variables and criteria as defined above

$$\frac{\partial \theta_1(x, Fo)}{\partial Fo} = \frac{\partial^2 \theta_1(x, Fo)}{\partial x^2} + \frac{1}{x}\frac{\partial \theta_1(x, Fo)}{\partial x} + \epsilon Ko \frac{\partial \theta_2(x, Fo)}{\partial Fo}, \tag{3}$$

and

$$\frac{\partial \theta_2(x, Fo)}{\partial Fo} = Lu\left[\frac{\partial^2 \theta_2(x, Fo)}{\partial x^2} + \frac{1}{x}\frac{\partial \theta_2(x, Fo)}{\partial x}\right] + Lu\, Pn\left[\frac{\partial^2 \theta_1(x, Fo)}{\partial x^2} + \frac{1}{x}\frac{\partial \theta_1(x, Fo)}{\partial x}\right] \tag{4}$$

$$0 < x < 1,\ Fo > 0.$$

The boundary conditions for the system of differential equations (3) and (4) are supposed to be

$$\theta_{1,x}(1, Fo) + A_1\theta_1(1, Fo) + B_1\theta_2(1, Fo) = \phi_1(Fo), \tag{5}$$

$$\theta_{2,x}(1, Fo) + A_2\theta_{1,x}(1, Fo) + B_2\theta_2(1, Fo) = \phi_2(Fo) \tag{6}$$

and

$$\theta_{i,x}(0, Fo) = 0, \tag{7}$$

where A_i and B_i are aggregate of known dimensionless thermophysical coefficients and $\phi_i(Fo)$ are prescribed fluxes to be determined by the experiment.

For the complete statement of the problem, we shall specify the initial conditions as

$$\theta_i(x, 0) = f_i(x), \tag{8}$$

where $f_i(x)$ are the given functions.

SOLUTION OF THE PROBLEM

Let us take an auxiliary problem in which the boundary conditions (5) and (6) are replaced by

$$\theta_{i,x}(1, Fo) = \chi_i(Fo). \tag{9}$$

We shall now determine the solution of this auxiliary problem with the aid of the conditions (7) and (8). For this purpose, we define a Hankel transform with respect to space variable x:

$$\bar{\theta}_i(p, Fo) = \int_0^1 x\theta_i(x, Fo)\, J_0(p\, x)\, dx, \tag{10}$$

where $p = p_n$ are the roots of the characteristic equation

$$J_1(p) = 0 \tag{11}$$

and a Laplace transform with respect to time variable Fo,

$$\hat{\theta}_i(p,s) = \int_0^\infty e^{-sFo}\theta_i(p,Fo)\,dFo. \quad (12)$$

The inversion formulae for (10) and (12) are:

$$\theta_i(x,Fo) = 2\int_0^1 x\theta_i(x,Fo)\,dFo + 2\sum_{n=1}^{\infty}\frac{J_0(p_n,x)}{J_0^2(p_n)}\bar{\theta}_i(p_n,Fo) \quad (13)$$

and

$$\theta_i(p,Fo) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sFo}\,\theta_i(p,s)\,ds, \quad (14)$$

where the integration is carried over the straight line $s = \sigma$ in the complex plane.

Simultaneous applications of the Hankel and Laplace transforms to the equations (3) and (4) yield a set of two simultaneous equations

$$s\hat{\theta}_1 - \bar{f}_1(p) = \hat{\chi}_1(s) - p^2\hat{\theta}_1 + \epsilon K_0(s\hat{\theta}_2 - \bar{f}_2(p)) \quad (15)$$

and

$$s\hat{\theta}_2 - \bar{f}_2(p) = Lu(\hat{\chi}_2(s) - p^2\hat{\theta}_2) + Lu\,Pn(\hat{\chi}_1(s) - p^2\hat{\theta}_1). \quad (16)$$

On solving these two equations, we find

$$\hat{\theta}_i(p,s) = \sum_{j,s=1}^{2}\left[A^i_{js}\frac{\bar{f}_s(p)}{s + V_j^2\,Lup^2} + B^i_{js}\frac{\hat{\chi}_s(s)}{s + V_j^2\,Lup^2}J_0(p)\right] \quad (17)$$

where

$$V_j^2 = \tfrac{1}{2}\left\{\left[1 + \frac{1}{Lu} + \epsilon Ko\,Pn\right] + (-1)^j \times\left(\left[1 + \frac{1}{Lu} + \epsilon KoPn\right]^2 - \frac{1}{Lu}\right)^{\frac{1}{2}}\right\} \quad (18)$$

and the constant coefficients are given by

$$A^1_{j1} = (-1)^j\frac{1 - V_j^2}{V_1^2 - V_2^2},\quad A^1_{j2} = (-1)^j\frac{1}{V_2^2 - V_1^2}\epsilon Ko;$$

$$B^1_{j1} = (-1)^j\frac{1 - V_j^2 - \epsilon KoLuPnV_j^2}{V_1^2 - V_2^2},$$

$$B^1_{j2} = (-1)^j\frac{1}{V_2^2 - V_1^2}\epsilon KoLuV_j^2;$$

$$A^2_{j1} = (-1)^j\frac{1}{V_2^2 - V_1^2}Pn,$$

$$A^2_{j2} = (-1)^j\frac{V_j^2 - \frac{1}{Lu} - \epsilon KoPn}{V_2^2 - V_1^2};$$

$$B^2_{j1} = (-1)^j\frac{V_j^2}{V_2^2 - V_1^2}LuPn,$$

$$B^2_{j2} = (-1)^j\frac{\frac{1}{Lu} - V_j^2}{V_2^2 - V_1^2}LuPn.$$

Applying the inversion formulae (13) and (14) the expressions for transfer potentials are obtained as

$$\theta_i(x,Fo) = \sum_{j,s=1}^{2}[A^i_{js}P_{js} + B^i_{js}Q_{js}], \quad (19)$$

where

$$P_{js} = 2\left[\int_0^1 xf_s(x)\,dx + \sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0^2(\mu_n)}\exp(-Lu\mu_n^2V_j^2Fo)\int_0^1 xf_s(x)J_0(\mu_n x)\,dx\right] \quad (20)$$

and

$$Q_{js} = 2\left[\int_0^{Fo}\chi_s(u)\,du + \sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\int_0^{Fo}\chi_s(u)\exp(-\mu_n^2V_j^2Lu\overline{Fo\text{-}u})\,du\right], \quad (21)$$

μ_n are the roots of the equation $J_1(\mu_n) = 0$.

The expressions (19) are the solutions for the auxiliary problem. In order to solve the present problem it is necessary to determine $\chi_i(Fo)$ from the original boundary conditions (5) and (6).

Substituting the values of $\partial\theta_i/\partial x$ and θ_i at $x = 1$ from the equations (9) and (19), we obtain

$$\chi_1(Fo) + 2 = \sum_{j,s=1}^{2} (A_1B_{js}^1 + B_1B_{js}^2)\left[\int_0^{Fo} \chi_s(u)\,du + \sum_{n=1}^{\infty} \frac{J_0(\mu_n x)}{J_0(\mu_n)} \int_0^{Fo} \exp[-\mu_n^2 V_j^2 Lu(Fo-u)]\,\chi_s(u)\,du\right] = \phi_1(Fo) - \sum_{j,s=1}^{2} (A_1A_{js}^1 + B_1A_{js}^2)P_{js} \quad (22)$$

and

$$A_2\chi_1(Fo) + \chi_2(Fo) + 2B_2 \sum_{j,s=1}^{2} B_{js}^2\left[\int_0^{Fo} \chi_s(u)\,du + \sum_{n=1}^{\infty} \frac{J_0(\mu_n x)}{J_0(\mu_n)} \int_0^{Fo} \exp[-\mu_n^2 Lu(Fo-u)]\chi_s(u)\,du\right] = \phi_2(Fo) - B_2 \sum_{j,s=1}^{2} A_{js}^2 P_{js}. \quad (23)$$

The integral equations (22) and (23) determine the values of the functions $\hat{\chi}_i(Fo)$ and by substituting the values of $\chi_s(u)$ in (21), we obtain the expressions for Q_{js} and hence the expressions for the transfer potentials are determined.

However, it is very difficult to solve the integral equations (22) and (23). We are going to determine an approximate solutions of $\chi_i(Fo)$ for small values of generalized time which may serve the needs of the engineering calculations.

Under the Laplace transformation, the integral equations (22) and (23) take the form.

$$\hat{\chi}_1(s) + 2 \sum_{j,s=1}^{2} (A_1B_{js}^1 + B_1B_{js}^2)\left(\frac{1}{s} + \sum_{n=1} \frac{J_0(\mu_n x)}{J_0(\mu_n)} \frac{1}{s + \mu_n V_j^2 Lu}\right)\hat{\chi}_s(s) = \hat{\vartheta}_1(s) \quad (24)$$

and

$$\hat{\chi}_2(s) + 2\hat{\chi}_1(s) + 2B_2 \sum_{j,s+1}^{2} B_{js}^2\left(\frac{1}{s} + \sum_{n=1}^{\infty} \frac{J_0(\mu_n x)}{J_0(\mu_n)} \frac{1}{s + \mu_n V_j^2 Lu}\right)\hat{\chi}_s(s) = \hat{\vartheta}_2(s) \quad (25)$$

where

$$\hat{\vartheta}_1(s) = \int_0^{\infty} e^{-sFo}\left\{\phi_1(Fo) - \sum_{j,s=1}^{2} (A_1A_{js}^1 + B_1A_{js}^1)P_{js}\right\} dFo \quad (26)$$

and

$$\hat{\vartheta}_2(s) = \int_0^{\infty} e^{-sFo}\left\{\phi_2(Fo) - B_2 \sum_{j,s=1}^{2} A_{js}^2 P_{js}\right\} dFo. \quad (27)$$

Equations (24) and (25) are two ordinary simultaneous equations in $\hat{\chi}_1(s)$ and $\hat{\chi}_2(s)$ and contain a series which converges ultimately. Now approximating the term

$$\frac{1}{s + \mu_n LuV_j^2} \approx \frac{1}{s},$$

we have

$$\hat{\chi}_1(s) + 2 \sum_{j,s=1}^{2} \left[(A_1B'_{js} + B_1B_{js}^1)\left(1 + \sum_{n=1}^{\infty} \frac{J_0(\mu_n x)}{J_0(\mu_n)} \frac{1}{s}\hat{\chi}_s(s)\right]\right. = \hat{\vartheta}_1(s) \quad (28)$$

and

$$\hat{\chi}_2(s) + A_2\hat{\chi}_1(s) + 2B_2 \sum_{j,s=1}^{\infty}$$

$$\times\left[B_{js}^2\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right)\frac{1}{s}\hat{\chi}_s(s)\right] = \hat{g}_2(S). \qquad (29)$$

Solving these two equations for $\hat{\chi}_1(s)$ and $\chi_2(s)$ and restricting to the terms of order $1/s$ only, we obtain

$$\hat{\chi}_1(s) = \left[\hat{g}_1(s)\left\{1+\frac{2B_2}{s}\sum_j B_{j2}^2 \times\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right)\right\} - \frac{2}{s}\hat{g}_2(s)\sum_{j=1}^{2} \times (A_1 B_{j2}^1 + B_1 B_{j2}^2) \times\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right)\right]\Big/\left(1+\frac{2\alpha}{s}\right) \qquad (30)$$

and

$$\hat{\chi}_2(s) = \left[-\hat{g}_1(s)\left\{A_2+\frac{2B_2}{s}\sum_{j=1}B_{j1}^2 \times\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right)\right\} + \hat{g}_2(s) \times\left\{1+\frac{2}{s}\sum_{j=1}^{2}(A_1 B_{j1}^1 + B_1 B_{j1}^2) \times\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right)\right\}\right]\Big/\left(1+\frac{2\alpha}{s}\right), \qquad (31)$$

where

$$\alpha = \sum_{j=1}^{2}(A_1 B_{j1}^1 + B_1 B_{j1}^2 + B_2 B_{j2}^2 - A_2 A_1 B_{j2}^1 - A_2 B_1 B_{j2}^2) \times\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right).$$

The inversion can be carried by expanding $(1 + 2\alpha/s)^{-1}$ and considering the term of order s^{-1} and s^0 only. The inverted expression of $\hat{\chi}_1(s)$ are

$$\chi_1(Fo) = g_1(Fo) - 2\alpha\int_0^{Fo} g_1(u)\,du + 2B_2\sum_{j=1}^{2}B_{j2}^2\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right) \times\int_0^{Fo} g_1(u)\,du - 2\sum_{j=1}^{2}(A_1 B_{j1}^2 + B_1 B_{j2}^2) \times\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right)\int_0^{Fo} g_2(u)\,du \qquad (32)$$

and

$$\chi_2(Fo) = -A_2 g_1(Fo) + g_2(Fo) - 2A_2\alpha\int_0^{Fo} g_1(u)\,du + 2\alpha\int_0^{Fo} g_2(u)\,du - 2B_2\sum_{j=1}^{2}B_{j1}^2\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right) \times\int_0^{Fo} g_1(u)\,du + 2\sum_{j=1}^{2}(A_1 B_{j1}^1 + B_1 B_{j1}^2) \times\left(1+\sum_{n=1}^{\infty}\frac{J_0(\mu_n x)}{J_0(\mu_n)}\right)\int_0^{Fo} g_2(u)\,du. \qquad (33)$$

Expressions (32) and (33) determine the auxiliary functions $\chi_i(Fo)$ and this after substitution in equation (21) gives out the value of the function Q_{js} which ultimately determines the function $\theta_i(x, Fo)$.

The boundary conditions referred in (5) and (6) are of most general type and include all types of the interaction law between solid body and gaseous medium. The suitably chosen values of the thermophysical coefficients A_i and B_i determine any type of the interaction law.

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UNE APPROCHE PAR UNE EQUATION INTEGRALE DU PROBLEME DE TRANSFERT THERMIQUE ET MASSIQUE DANS UN CYLINDRE INFINI

Résumé—Les auteurs ont développé une méthode d'équations intégrales pour obtenir les potentiels de transfert de chaleur et de masse dans un corps cylindrique unidimensionnel sous l'influence de conditions aux limites de type le plus général. On a recherché pour le systéme une solution approchée, applicable pour des petites valeurs du temps généralisé.

LÖSUNG DES WÄRME- UND STOFFÜBERGANGSPROBLEMS IN EINEM UNENDLICH LANGEN ZYLINDER MIT HILFE VON INTEGRALGLEICHUNGEN

Zusammenfassung—Die Autoren haben eine auf Integralgleichungen basierende Methode entwickelt, mit der sich die Übertragungspotentiale für Wärme und Stoff in einem eindimensionalen, unendlich langen Zylinder unter dem Einfluss des allgemeinsten Typs von Randbedingungen gewinnen lassen. Eine Näherungslösung, gültig für kleine Werte des generalisierten Zeitparameters, ist ebenfalls für das System ausgearbeitet worden.

ИНТЕГРАЛЬНЫЙ МЕТОД РЕШЕНИЯ ЗАДАЧ ТЕПЛО- И МАССО-ОБМЕНА В БЕСКОНЕЧНОМ ЦИЛИНДРЕ

Аннотация—В статье приводится метод решения интегральных уравнений, разработанный для определения потенциалов переноса тепла и массы в одномерном бесконечном цилиндре при самых общих граничных условиях. Получено приближенное решение, применимое для малых значений безразмерного времени.